JOURNAL OF APPROXIMATION THEORY **42**, 64–69 (1984)

Uniqueness Theorems for Entire Functions of Exponential Type

A. M. TREMBINSKA

Department of Mathematics, Northwestern University, Evanston, Illinois 60201, U.S.A.

Communicated by Oved Shisha

Received October 18, 1983

Carlson's theorem [1, p. 153] states that an entire function of exponential type τ less than π must vanish identically if it vanishes at the integers. Let $f(z) = u(z) + iv(z)(u, v$ real). It is not enough to assume that $u(m) = 0$, $m = 0, \pm 1, \pm 2,...$, in order to conclude $f(z) \equiv 0$, but it is known to be sufficient to have both $u(m) = 0$ and $u(m+i) = 0$, $m = 0, \pm 1, \pm 2,...$ (see [2]). Here I show that if $f(z)$ satisfies mild restrictions on its growth on the real axis and $u(m) = 0$, then $f(z) \equiv 0$ provided that $u(m+i) = 0$ or $v(m+i) = 0$, except on a set of integers of density less than $1 - (\tau/\pi)$. Hence I consider a somewhat more restricted class of functions than in [2], but obtain much sharper uniqueness theorems.

I thank Professor R. P. Boas for suggesting the problem.

THEOREM 1. *Let $f(z)$ be an entire function of exponential type τ less than π. Let $f(m) = u(m) + iv(m)$, $u(m) = 0$, $m = 0, \pm 1, \pm 2,...$, $|f(m)| < M$, $m = 0, \pm 1, \pm 2,...$, and $\sum_{-\infty}^{\infty} |v(m)| < \infty$. If $u(m+i) = 0$ except on a set of points of density less than $1 - (\tau/\pi)$, then $f(z) \equiv 0$.*

Proof. Since $f(z)$ is of exponential type τ less than π and is bounded at the integers, it follows from Cartwright's theorem [1, p. 180] that $f(x)$ is bounded for all real x. Hence $f(z)$ has the representation

$$f(z) = f'(0) \frac{\sin \pi z}{\pi} + f(0) \frac{\sin \pi z}{\pi z} + \frac{z \sin \pi z}{\pi} \sum_{n \neq 0} \frac{(-1)^n f(n)}{n(z-n)}$$

(see [1], p. 221). Set $z = m + i$, keeping in mind that $f(m) = iv(m)$ and $\sin \pi(m+i) = i(-1)^m \sinh \pi$, to obtain

0021-9045/84 \$3.00

$$f(m+i) = v'(0)(-1)^{m+1}\frac{\sinh\pi}{\pi} + iu'(0)(-1)^m\frac{\sinh\pi}{\pi}$$
$$+ v(0)(-1)^{m+1}\frac{m\sinh\pi}{\pi(m^2+1)} - iv(0)(-1)^{m+1}\frac{\sinh\pi}{\pi(m^2+1)}$$
$$+ m(-1)^{m+1}\frac{\sinh\pi}{\pi}\sum_{n\neq 0}\frac{(-1)^n v(n)}{n(m-n+i)}$$
$$+ i(-1)^{m+1}\frac{\sinh\pi}{\pi}\sum_{n\neq 0}\frac{(-1)^n v(n)}{n(m-n+i)}.$$

Multiply both sums on the right of the above equation by $(m-n)-i/(m-n)-i$ to obtain

$$f(m+i) = v'(0)(-1)^{m+1}\frac{\sinh\pi}{\pi} + iu'(0)(-1)^m\frac{\sinh\pi}{\pi}$$
$$+ v(0)(-1)^{m+1}\frac{m\sinh\pi}{\pi(m^2+1)} - iv(0)(-1)^{m+1}\frac{\sinh\pi}{\pi(m^2+1)}$$
$$+\frac{m\sinh\pi}{\pi}\sum_{n\neq 0}\frac{(-1)^{m-n+1}(m-n)\,v(n)}{n[(m-n)^2+1]}$$
$$-\frac{im\sinh\pi}{\pi}\sum_{n\neq 0}\frac{(-1)^{m-n+1}v(n)}{n[(m-n)^2+1]}$$
$$+\frac{i\sinh\pi}{\pi}\sum_{n\neq 0}\frac{(-1)^{m-n+1}(m-n)\,v(n)}{n[(m-n)^2+1]}$$
$$+\frac{\sinh\pi}{\pi}\sum_{n\neq 0}\frac{(-1)^{m-n+1}v(n)}{n[(m-n)^2+1]}.$$

We then have

$$\operatorname{Re} f(m+i) = u(m+i) = \frac{-\sinh\pi}{\pi}\Bigg[v'(0)(-1)^m + v(0)(-1)^m\frac{m}{m^2+1}$$
$$+ m\sum_{n\neq 0}\frac{(-1)^{m-n}(m-n)\,v(n)}{n[(m-n)^2+1]} + \sum_{n\neq 0}\frac{(-1)^{m-n}v(n)}{n[(m-n)^2+1]}\Bigg],$$

$$\operatorname{Im} f(m+i) = v(m+i) = \frac{-\sinh\pi}{\pi}\Bigg[u'(0)(-1)^{m-1} - v(0)\frac{(-1)^m}{m^2+1}$$
$$+\sum_{n\neq 0}\frac{(-1)^{m-n}(m-n)\,v(n)}{n[(m-n)^2+1]} - m\sum_{n\neq 0}\frac{(-1)^{m-n}v(n)}{n[(m-n)^2+1]}\Bigg].$$

By [4, p. 13] with $a=b=1$, $l=\pi$,

$$i \sum_{n\neq 0} \frac{(-1)^{m-n}(m-n)}{(m-n)^2+1} \sin(m-n)\, x = -i \frac{\pi \sinh x}{\sinh \pi}, \quad -\pi < x < \pi.$$

Since the convolution of the Fourier coefficients of two functions yields the Fourier coefficients of the product of the functions (see [3], p. 23) we have that

$$\sum_{n\neq 0} \frac{(-1)^{m-n}(m-n)\, v(n)}{n[(m-n)^2+1]}$$

is the mth Fourier coefficient of

$$\frac{-i\pi \sinh x}{\sinh \pi} \sum_{n\neq 0} \frac{v(n)}{n} e^{inx}, \qquad -\pi < x < \pi.$$

Similarly, by [4, p. 12], we have, for $a=b=1$, $l=\pi$,

$$\sum_{-\infty}^{\infty} \frac{(-1)^{m-n}}{(m-n)^2+1} \cos(m-n)\, x = \frac{\pi \cosh x}{\sinh \pi}, \qquad -\pi \leqslant x \leqslant \pi,$$

so that

$$\sum_{n\neq 0} \frac{(-1)^{m-n} v(n)}{n[(m-n)^2+1]}$$

is the mth Fourier coefficient of

$$\frac{\pi \cosh x}{\sinh \pi} \sum_{n\neq 0} \frac{v(n)}{n} e^{inx}, \qquad -\pi \leqslant x \leqslant \pi.$$

Also, $v(0)(-1)^m m/(m^2+1)$ is the mth Fourier coefficient of

$$iv(0) \sum_{-\infty}^{\infty} \frac{(-1)^m m}{m^2+1} \sin mx = \frac{-\, iv(0)\, \pi \sinh x}{\sinh \pi}, \qquad -\pi < x < \pi.$$

Since $a_m = (-1)^m$, $m = 0, \pm 1, \pm 2, \ldots$, are not the Fourier coefficients of any absolutely integrable function, we must have $v'(0) = 0$.

Hence,

$$u(m+i) = \frac{-\sinh \pi}{\pi} \left[v(0)(-1)^m \frac{m}{m^2+1} + m \sum_{n\neq 0} \frac{(-1)^{m-n}(m-n)\, v(n)}{n[(m-n)^2+1]} + \sum_{n\neq 0} \frac{(-1)^{m-n} v(n)}{n[(m-n)^2+1]} \right]$$

is the mth Fourier coefficient of

$$F(x)=\frac{-\sinh\pi}{\pi}\left[\frac{-iv(0)\,\pi\sinh x}{\sinh\pi}+\frac{1}{i}\frac{d}{dx}\left\{\frac{-i\pi\sinh x}{\sinh\pi}\sum_{n\neq 0}\frac{v(n)}{n}e^{inx}\right\}\right.$$
$$\left.+\frac{\pi\cosh x}{\sinh\pi}\sum_{n\neq 0}\frac{v(n)}{n}e^{inx}\right],\qquad -\pi<x<\pi.$$

Simplifying, we find that $u(m+i)$ is the mth Fourier coefficient of

$$F(x)=iv(0)\sinh x+\cosh x\sum_{n\neq 0}\frac{v(n)}{n}e^{inx}$$
$$+\,i\sinh x\sum_{n\neq 0}v(n)\,e^{inx}-\cosh x\sum_{n\neq 0}\frac{v(n)}{n}e^{inx},$$
$$F(x)=i\sinh x\sum_{-\infty}^{\infty}v(n)\,e^{inx},\qquad -\pi<x<\pi.$$

The hypothesis implies that $\int_{-\tau}^{\tau}F(x)\,e^{i\lambda_n x}=0$ for a sequence of integers λ_n of density greater than τ/π. By [1, p. 236] (with a change of variable) $F(x)=i\sinh x\sum_{-\infty}^{\infty}v(n)\,e^{inx}$ vanishes for almost all x in $(-\pi,\pi)$. Since $\sinh x=0$ if and only if $x=0$, we conclude $v(n)\equiv 0$. Hence $f(n)=u(n)+iv(n)=0$ for all n, and by Carlson's theorem $f(z)\equiv 0$.

THEOREM 2. *With the hypotheses of Theorem* 1, *if* $v(m+i)=0$ *except on a set of points of density less than* $1-(\tau/\pi)$, *then* $f(z)\equiv 0$.

Proof. Recall that

$$v(m+i)=\frac{-\sinh\pi}{\pi}\left[u'(0)(-1)^{m+1}-v(0)\frac{(-1)^m}{m^2+1}\right.$$
$$+\sum_{n\neq 0}\frac{(-1)^{m-n}(m-n)\,v(n)}{n[(m-n)^2+1]}$$
$$\left.-\,m\sum_{n\neq 0}\frac{(-1)^{m-n}v(n)}{n[(m-n)^2+1]}\right].$$

Proceeding as in the proof of Theorem 1, we conclude that $v(m+i)$ is the mth Fourier coefficient of

$$G(x)=\frac{-\sinh\pi}{\pi}\left[-v(0)\frac{\pi\cosh x}{\sinh\pi}-i\frac{\pi\sinh x}{\sinh\pi}\sum_{n\neq 0}\frac{v(n)}{n}e^{inx}\right.$$
$$\left.-\frac{1}{i}\frac{d}{dx}\left\{\frac{\pi\cosh x}{\sinh\pi}\sum_{n\neq 0}\frac{v(n)}{n}e^{inx}\right\}\right],\qquad -\pi<x<\pi.$$

Simplifying, we find that $v(m+i)$ is the mth Fourier coefficient of

$$G(x) = v(0)\cosh x + i\sinh x \sum_{n\neq 0} \frac{v(n)}{n} e^{inx} - i\sinh x \sum_{n\neq 0} \frac{v(n)}{n} e^{inx}$$

$$+ \cosh x \sum_{n\neq 0} v(n)\, e^{inx},$$

$$G(x) = \cosh x \sum_{-\infty}^{\infty} v(n)\, e^{inx}, \qquad -\pi < x < \pi.$$

Now continue as in the proof of Theorem 1, with $F(x)$ replaced with $G(x)$.

The type π is critical, as is shown in Theorem 3.

THEOREM 3. *There exist entire functions $f(z)$ of exact type π with $f(m) = u(m) + iv(m)$, $u(m) = 0$, $m = 0, \pm 1, \pm 2,\ldots$, $|f(m)| < M$, $m = 0, \pm 1, \pm 2,\ldots$, and $\sum_{-\infty}^{\infty} |v(m)| < \infty$, such that either* (i) $v(m+i) = 0$, *or* (ii) $u(m+i) = 0$, *in either case except on a set of integers $\{m_n\}$ of density zero, but $f(z) \not\equiv 0$.*

Proof. (i) Consider

$$f(z) = i\int_{-\pi}^{\pi} \frac{\sin zt}{\cosh t} \sum_n b_n \sin m_n t\, dt,$$

where $b_n \geqslant 0$, $\sum_n b_n m_n^2 < \infty$, and the set of integers $\{m_n\}$ has density zero. By [1, p. 108] $f(z)$ is of exact type π. Since the integrand is real on the real axis, $f(m)$ is pure imaginary, and $u(m) = 0$ for all m. Also, $|\sin mt|/\cosh t \leqslant 1$, so that $|f(m)| \leqslant \sum_n |b_n| \int_{-\pi}^{\pi} |\sin m_n t|\, dt \leqslant 2\pi \sum_n b_n < \infty$, and $|f(m)|$ is bounded for all m. Integrating $f(m)$ by parts twice, we find that

$$f(m) = \frac{i}{m^2} \int_{-\pi}^{\pi} \frac{\sin mt}{\cosh t} \sum_n b_n \sin m_n t\, dt$$

$$+ \frac{i}{m^2} \int_{-\pi}^{\pi} \frac{\sin mt}{\cosh t} \sum_n b_n m_n^2 \sin m_n t\, dt$$

$$+ \frac{2i}{m^2} \int_{-\pi}^{\pi} \frac{\sin mt \sinh t}{\cosh^2 t} \sum_n b_n m_n \cos m_n t\, dt$$

$$- \frac{2i}{m^2} \int_{-\pi}^{\pi} \frac{\sin mt \sinh^2 t}{\cosh^3 t} \sum_n b_n \sin m_n t\, dt.$$

Since each integrand on the right of the above equation is a bounded function on $[-\pi, \pi]$, $|f(m)| = |v(m)| \leqslant M/m^2$, M a constant. Hence

$\sum_{-\infty}^{\infty} |v(m)| \leqslant M \sum_{m \neq 0} 1/m^2 < \infty$. A simple computation shows that $v(m+i) = \int_{-\pi}^{\pi} \sin mt \sum_n b_n \sin m_n t \, dt$, so that $v(m+i) = 0$, $m \neq m_n$, and $v(m+i) \neq 0$, $m = m_n$.

(ii) Consider

$$f(z) = i \int_{-\pi}^{\pi} \frac{\sin zt}{\sinh t} \sum_n b_n (1 - \cos m_n t) \, dt,$$

where $\sum_n b_n = 0$, $\sum_n |b_n| m_n^2 < \infty$, and the set of integers $\{m_n\}$ has density zero. Proceed as in the proof of (i) and conclude that $f(z)$ is of exact type π, $u(m) = 0$ for all m, and $|f(m)|$ is bounded for all m. Integrate $f(m)$ by parts twice and obtain

$$\begin{aligned} f(m) = & \frac{-2i}{m^2} \int_0^{\pi} \frac{\sin mt}{\sinh t} \sum_n b_n m_n^2 \cos m_n t \, dt \\ & + \frac{4i}{m^2} \int_0^{\pi} \frac{\sin mt \cosh t}{\sinh^2 t} \sum_n b_n m_n \sin m_n t \, dt \\ & + \frac{2i}{m^2} \int_0^{\pi} \frac{\sin mt}{\sinh t} \sum_n b_n (1 - \cos m_n t) \, dt \\ & - \frac{4i}{m^2} \int_0^{\pi} \frac{\sin mt \cosh^2 t}{\sinh^3 t} \sum_n b_n (1 - \cos m_n t) \, dt. \end{aligned}$$

Since each integrand on the right of the above equation is a bounded function on $[0, \pi]$, we once again conclude $\sum_{-\infty}^{\infty} |v(m)| < \infty$. Since $u(m+i) = \int_{-\pi}^{\pi} \cos mt \sum_n b_n \cos m_n t \, dt$, $u(m+i) = 0$, $m \neq m_n$, and $u(m+i) \neq 0$, $m = m_n$.

References

1. R. P. Boas, Jr., "Entire Functions," Academic Press, New York, 1954.
2. R. P. Boas, Jr., A uniqueness theorem for harmonic functions, *J. Approx. Theory* 5 (1972), 425–427.
3. G. H. Hardy and W. W. Rogosinski, "Fourier Series," Cambridge Univ. Press, Cambridge, England, 1944.
4. F. Oberhettinger, "Fourier Expansions," Academic Press, New York, 1973.